第五节 复合函数微分法与隐函数微分法

一 链式法则

二 全微分形式不变性

三 隐函数微分法

一、链式法则

1 复合函数的中间变量为一元函数的情形

定理 1 如果函数$u=\varphi(t)$及$v=\psi(t)$都在点t可导，函数$z=f(u,v)$在对应点(u,v)具有连续偏导数，则复合函数$z=f[\varphi(t),\psi(t)]$在对应点t可导，且其导数可用下列公式计算：

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}$$

证 设t获得增量Δt

则$\Delta u=\varphi(t+\Delta t)-\varphi(t)$，$\Delta v=\psi(t+\Delta t)-\psi(t)$

由于函数$z=f(u,v)$在点(u,v)有连续偏导数

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+\varepsilon_1\Delta u+\varepsilon_2\Delta v,$$

当$\Delta u\to 0,\Delta v\to 0$时，$\varepsilon_1\to 0,\varepsilon_2\to 0$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta t}+\frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta t}+\varepsilon_1\frac{\Delta u}{\Delta t}+\varepsilon_2\frac{\Delta v}{\Delta t}$$

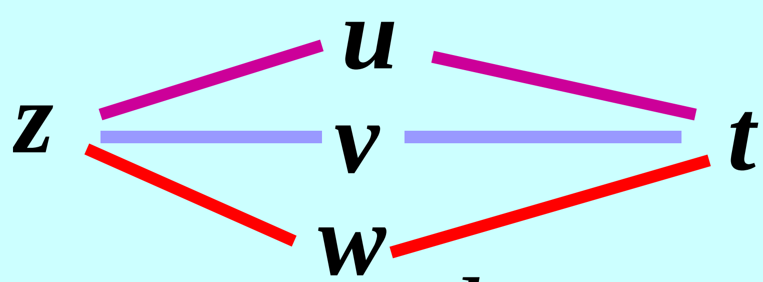
当$\Delta t\to 0$时， $\Delta u\to 0,\Delta v\to 0$

$$\frac{\Delta u}{\Delta t}\to\frac{du}{dt},\qquad \frac{\Delta v}{\Delta t}\to\frac{dv}{dt},$$

$$\frac{dz}{dt}=\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}.$$

上定理的结论可推广到中间变量多于两个的情况.

如 $$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}$$



以上公式中的导数 $\frac{dz}{dt}$ 称为全导数.

2 复合函数的中间变量为多元函数的情形

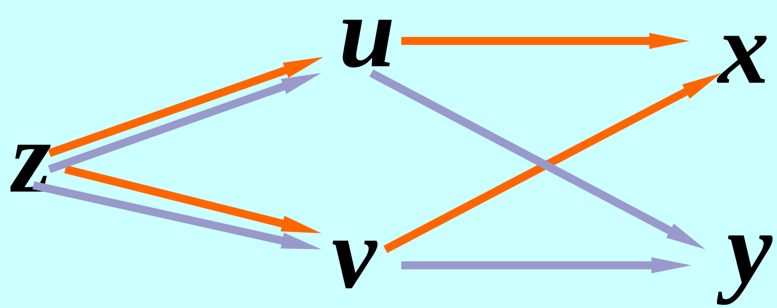
$$z = f[\phi(x,y),\psi(x,y)].$$

定理2

如果$u=\varphi(x,y)$及$v=\psi(x,y)$都在点(x,y)具有对x和y的偏导数且函数$z=f(u,v)$在对应点(u,v)具有连续偏导数，则复合函数$z=f[\varphi(x,y),\psi(x,y)]$在对应点$(x,y)$的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

链式法则如图示

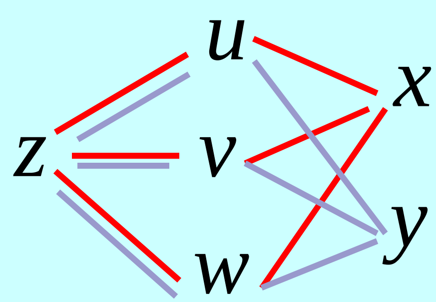


$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

类似地再推广，设$u=\varphi(x,y)$、$v=\psi(x,y)$、$w=\omega(x,y)$都在点(x,y)具有对x和y的偏导数，复合函数$z=f[\varphi(x,y),\psi(x,y),\omega(x,y)]$在对应点$(x,y)$两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x}$$



$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y}$$

特殊地 $z=f(u,x,y)$ 其中 $u=\phi(x,y)$

即 $z=f[\phi(x,y),x,y]$, 令 $v=x$, $w=y$,

$$\frac{\partial v}{\partial x}=1,\quad \frac{\partial w}{\partial x}=0,\quad \frac{\partial v}{\partial y}=0,\quad \frac{\partial w}{\partial y}=1.$$

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial y}.$$

区别类似

两者的区别

把复合函数 $z=f[\phi(x,y),x,y]$ 中的 y 看作不变而对 x 的偏导数

把 $z=f(u,x,y)$ 中的 u 及 y 看作不变而对 x 的偏导数

例 1 设 $z = e^u \sin v$，而 $u = xy$， $v = x + y$

求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1 = e^u (y \sin v + \cos v),$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= e^u \sin v \cdot x + e^u \cos v \cdot 1 = e^u (x \sin v + \cos v).$$

例 2　设$z = uv + \sin t$，而$u = e^t$，$v = \cos t$，

求全导数$\dfrac{dz}{dt}$.

解

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial t}$$

$$= ve^t - u\sin t + \cos t$$

$$= e^t\cos t - e^t\sin t + \cos t$$

$$= e^t(\cos t - \sin t) + \cos t.$$

例3　设 $w = f(x+y+z, xyz)$，f 具有二阶连续偏导数，求 $\frac{\partial w}{\partial x}$ 和 $\frac{\partial^2 w}{\partial x^2}$.

解　令 $u = x+y+z,\ v = xyz;$

记 $f_1' = \frac{\partial f(u,v)}{\partial u}$，$f_{12}'' = \frac{\partial^2 f(u,v)}{\partial u \partial v}$，

同理有 f_2'，f_{11}''，f_{22}''.

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} = f_1' + yzf_2';$$

$$\frac{\partial^2 w}{\partial x \partial z} = \frac{\partial}{\partial z}(f_1' + yzf_2') = \frac{\partial f_1'}{\partial z} + yf_2' + yz\frac{\partial f_2'}{\partial z};$$

$$\frac{\partial f_1'}{\partial z} = \frac{\partial f_1'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_1'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{11}'' + xyf_{12}'';$$

$$\frac{\partial f_2'}{\partial z} = \frac{\partial f_2'}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial f_2'}{\partial v} \cdot \frac{\partial v}{\partial z} = f_{21}'' + xyf_{22}'';$$

于是 $\dfrac{\partial^2 w}{\partial x \partial z} = f_{11}'' + xyf_{12}'' + yf_2' + yz(f_{21}'' + xyf_{22}'')$

$$= f_{11}'' + y(x+z)f_{12}'' + xy^2 z f_{22}'' + yf_2'.$$

二、全微分形式不变性

设函数$z=f(u,v)$具有连续偏导数，则有全微分

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv;$$当 $u=\varphi(x,y),\ v=\psi(x,y),$

时，有 $$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

全微分形式不变形的实质：

无论 z 是自变量(u,v)的函数或中间变量(u,v)的函数，它的全微分形式是一样的.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$= \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy$$

$$= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right) + \frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)$$

$$= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

例4　已知 $e^{-xy}-2z+e^{z}=0$，求 $\frac{\partial z}{\partial x}$ 和 $\frac{\partial z}{\partial y}$.

解　$\Theta\ d(e^{-xy}-2z+e^{z})=0,$

$$\therefore e^{-xy}d(-xy)-2dz+e^{z}dz=0,$$

$$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$$

$$dz=\frac{ye^{-xy}}{(e^{z}-2)}dx+\frac{xe^{-xy}}{(e^{z}-2)}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$

作业：P282

1, 3, 4, 6(2), 7

三 隐函数的微分方法

定理1. 设函数 $F(x,y)$在点 $P(x_0,y_0)$ 的某一邻域内满足

① 具有连续的偏导数;

② $F(x_0, y_0) = 0$;

③ $F_y(x_0, y_0) \neq 0$

则方程 $F(x,y)=0$ 在点 x_0 的**某邻域内**可唯一确定一个单值连续函数 $y = f(x)$, 满足条件 $y_0 = f(x_0)$, 并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} \quad \text{(隐函数求导公式)}$$

设 $y=f(x)$ 为方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,f(x))\equiv 0$$

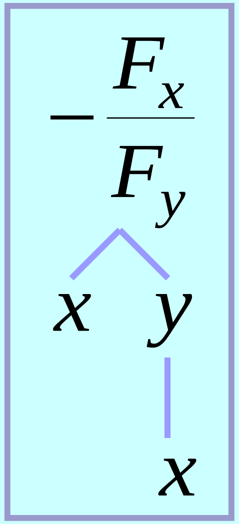
↓ 两边对 x 求导

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\equiv 0$$

↓ 在$(x_0\,,y_0)$的某邻域内 $F_y\neq 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$

若$F(x,y)$的二阶偏导数也都连续, 则还有



二阶导数：

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}=\frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right)+\frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=-\frac{F_{xx}F_y-F_{yx}F_x}{F_y^2}-\frac{F_{xy}F_y-F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$=-\frac{F_{xx}F_y{}^2-2F_{xy}F_xF_y+F_{yy}F_x{}^2}{F_y^3}$$

例1. 验证方程 $\sin y + e^x - xy - 1 = 0$ 在点(0,0)某邻域可确定一个单值可导隐函数 $y = f(x)$,并求

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0},\quad \left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}$$

解: 令 $F(x, y) = \sin y + e^x - xy - 1$, 则

① $F_x = e^x - y,\ F_y = \cos y - x$ 连续 ,

② $F(0,0) = 0,$

③ $F_y(0,0) = 1 \neq 0$

由 定理1 可知, 在 $x = 0$ 的某邻域内方程存在单值可导的隐函数 $y = f(x)$, 且

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=-\left.\frac{F_x}{F_y}\right|_{x=0}=-\left.\frac{e^x-y}{\cos y-x}\right|_{x=0,\,y=0}=-1$$

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0}$$

$$=-\left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{e^x-y}{\cos y-x}\right)\right|_{x=0,\ y=0,\ y'=-1}$$

$$=-\left.\frac{(e^x-y')(\cos y-x)-(e^x-y)(-\sin y\cdot y'-1)}{(\cos y-x)^2}\right|_{\substack{x=0\\ y=0\\ y'=-1}}$$

$$=-3$$

导数的另一求法 —— 利用隐函数求导

$$\sin y + e^x - xy - 1 = 0, \quad y = y(x)$$

↓ 两边对 x 求导

$$\cos y \cdot y' + e^x - y - xy' = 0$$

→ $y'\big|_{x=0} = -\dfrac{e^x - y}{\cos y - x}\Big|_{(0,0)} = -1$

↓ 两边再对 x 求导

$$-\sin y \cdot (y')^2 + \cos y \cdot y'' + e^x - y' - y' - xy'' = 0$$

↓ 令 $x = 0$，注意此时 $y = 0,\ y' = -1$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\Big|_{x=0} = -3$$

定理2 . 若函数 $F(x,y,z)$ 满足:

① 在点 $P(x_0,y_0,z_0)$ 的某邻域内具有**连续偏导数**,

② $F(x_0\,,y_0,z_0)=0$

③ $F_z(x_0\,,y_0\,,z_0)\neq 0$

则方程 $F(x,y,z)=0$ 在点 $(x_0\,,y_0)$ 某一邻域内可唯一确定一个单值连续函数 $z=f(x\,,y)$, 满足 $z_0=f(x_0\,,y_0)$, 并有连续偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

设 $z=f(x,y)$ 是方程 $F(x,y)=0$ 所确定的隐函数，则

$$F(x,y,f(x,y))\equiv 0$$

↓ 两边对 x 求偏导

$$F_x+F_z\frac{\partial z}{\partial x}\equiv 0$$

↓ 在 (x_0,y_0,z_0) 的某邻域内 $F_z\neq 0$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$$

同样可得 $$\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

例2. 设 $x^2+y^2+z^2-4z=0$, 求 $\dfrac{\partial^2 z}{\partial x^2}$.

解法1 利用隐函数求导

$$2x+2z\frac{\partial z}{\partial x}-4\frac{\partial z}{\partial x}=0 \longrightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z}$$

再对 x 求导

$$2+2\left(\frac{\partial z}{\partial x}\right)^2+2z\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x^2}=0$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{1+\left(\frac{\partial z}{\partial x}\right)^2}{2-z}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

解法2 利用公式

设 $F(x,y,z)=x^2+y^2+z^2-4z$

则 $F_x=2x,\ F_z=2z-4$

$$\therefore \quad \frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{x}{z-2}=\frac{x}{2-z}$$

两边对 x 求偏导

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{x}{2-z}\right)=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)^2+x^2}{(2-z)^3}$$

例3. 设$F(x,y)$具有连续偏导数, 已知方程 $F(\frac{x}{z},\frac{y}{z})=0$, 求 $\mathrm{d}z$.

解法1 利用偏导数公式. 设 $z=f(x,y)$ 是由方程 $F(\frac{x}{z},\frac{y}{z})=0$ 确定的隐函数, 则

$$\frac{\partial z}{\partial x}=-\frac{F_1'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_1'}{xF_1'+yF_2'}$$

$$\frac{\partial z}{\partial y}=-\frac{F_2'\cdot\frac{1}{z}}{F_1'\cdot(-\frac{x}{z^2})+F_2'\cdot(-\frac{y}{z^2})}=\frac{zF_2'}{xF_1'+yF_2'}$$

故 $$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

解法2 微分法. 对方程两边求微分:

$$F(\frac{x}{z},\frac{y}{z})=0$$

$$F_1'\cdot \mathrm{d}(\frac{x}{z})+F_2'\cdot \mathrm{d}(\frac{y}{z})=0$$

$$F_1'\cdot(\frac{z\mathrm{d}x-x\mathrm{d}z}{z^2})+F_2'\cdot(\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2})=0$$

$$\frac{xF_1'+yF_2'}{z^2}\mathrm{d}z=\frac{F_1'\mathrm{d}x+F_2'\mathrm{d}y}{z}$$

$$\mathrm{d}z=\frac{z}{xF_1'+yF_2'}(F_1'\mathrm{d}x+F_2'\mathrm{d}y)$$

例4　设 $z=f(x,y)$ 是方程 $F(x+y+z,x^2+y^2+z^2)=0$ 所确定的隐函数，求 $\dfrac{\partial z}{\partial x},\dfrac{\partial^2 z}{\partial x\partial y}$.

作业：

- P282 10, 11, 12