第二节 偏导数

一 偏导数的定义及其计算法

二 高阶偏导数

一、偏导数的定义及其计算法

偏增量：

设函数$z = f(x, y)$在点 $P(x, y)$ 的某邻域内有定义，$P'(x+\Delta x, y)$ 为该邻域内的点.

称函数的增量 $f(x+\Delta x, y) - f(x, y)$为关于 x 的偏增量，称函数的增量 $f(x, y) - f(x, y+\Delta y)$ 为 y 关于的偏增量.

定义 设函数$z=f(x,y)$在点(x_0,y_0)的某一邻域内有定义，当y固定在y_0而x在x_0处有增量Δx时，相应地函数有偏增量

$$f(x_0+\Delta x,y_0)-f(x_0,y_0)$$

如果$\lim\limits_{\Delta x\to 0}\dfrac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}$存在，则称此极限为函数$z=f(x,y)$在点$(x_0,y_0)$处对$x$的偏导数，记为

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}},\quad \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\ y=y_0}},\quad \left.Z_x\right|_{\substack{x=x_0\\ y=y_0}} \text{ 或 } f_x(x_0,y_0).$$

同理函数 $z = f(x, y)$ 在点 (x_0, y_0) 处对 y 的偏导数，可定义为

$$\lim_{\Delta y\to 0}\frac{f(x_0, y_0+\Delta y)-f(x_0, y_0)}{\Delta y}$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}}\quad \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\ y=y_0}}\quad \left.Z_y\right|_{\substack{x=x_0\\ y=y_0}} \text{ 或 } f_y(x_0,y_0)$$

如果函数 $z = f(x, y)$在区域 D内任一点 (x, y) 处对 x 的偏导数都存在，那么这个偏导数就是 x, y 的函数，它就称为函数 $z = f(x, y)$对自变量 x的偏导数，

记作 $\dfrac{\partial z}{\partial x}$ ， $\dfrac{\partial f}{\partial x}$ ， Z_x 或 $f_x(x, y)$.

类似可定义函数 $z = f(x, y)$对自变量 y的偏导数，记作 $\dfrac{\partial z}{\partial y}$ ， $\dfrac{\partial f}{\partial y}$ ， Z_y 或 $f_y(x, y)$.

偏导数的概念可以推广到二元以上函数

如三元函数 $u = f(x,y,z)$ 在 (x,y,z)处

$$f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y,z) - f(x,y,z)}{\Delta x},$$

$$f_y(x,y,z) = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y,z) - f(x,y,z)}{\Delta y},$$

$$f_z(x,y,z) = \lim_{\Delta z \to 0} \frac{f(x,y,z+\Delta z) - f(x,y,z)}{\Delta z}.$$

例 1 求 $z = x^2 + 3xy + y^2$ 在点 $(1,2)$ 处的偏导数.

解 $\dfrac{\partial z}{\partial x} = 2x + 3y$; $\dfrac{\partial z}{\partial y} = 3x + 2y$.

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=2}} = 2\times1+3\times2=8,$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=2}} = 3\times1+2\times2=7.$$

例 2 设 $z = x^y (x > 0, x \neq 1)$

求证：$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z$$

证 $$\frac{\partial z}{\partial x} = yx^{y-1} \qquad \frac{\partial z}{\partial y} = x^y \ln x$$

$$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = \frac{x}{y} yx^{x-1} + \frac{1}{\ln x} x^y \ln x$$

$$= x^y + x^y = 2z$$

例 3　设 $z=\arcsin\dfrac{x}{\sqrt{x^2+y^2}}$，求 $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$.

解

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)'_x$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{y^2}{\sqrt{(x^2+y^2)^3}}\qquad (\sqrt{y^2}=|y|)$$

$$=\frac{|y|}{x^2+y^2}.$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\dfrac{x^2}{x^2+y^2}}}\cdot\left(\frac{x}{\sqrt{x^2+y^2}}\right)_y'$$

$$=\frac{\sqrt{x^2+y^2}}{|y|}\cdot\frac{(-xy)}{\sqrt{(x^2+y^2)^3}}$$

$$=-\frac{x}{x^2+y^2}\operatorname{sgn}\frac{1}{y}\qquad (y\neq 0)$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x\neq 0\\ y=0}}$$ 不存在.

例4 设 $z = f(x, y) = \sqrt{|xy|}$，求 $f_x(0, 0)$, $f_y(0, 0)$.

解

$$f_x(0,0) = \lim_{x\to 0} \frac{\sqrt{|x \cdot 0|} - 0}{x} = 0$$

$$f_y(0,0) = \lim_{y\to 0} \frac{\sqrt{|0 \cdot y|} - 0}{y} = 0$$

注意：1）求分界点、不连续点处的偏导数要用定义求；

2） 偏导数 $\frac{\partial z}{\partial x}$ 是一个整体记号，不能拆分.

3） 多元函数计算对某变量偏导时，将其他变量看作常数即可；

偏导数存在与连续的关系

一元函数在某点可导 $\rightarrow$ 连续

多元函数在某点偏导数存在，函数是否连续？

例如， 函数 $f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$

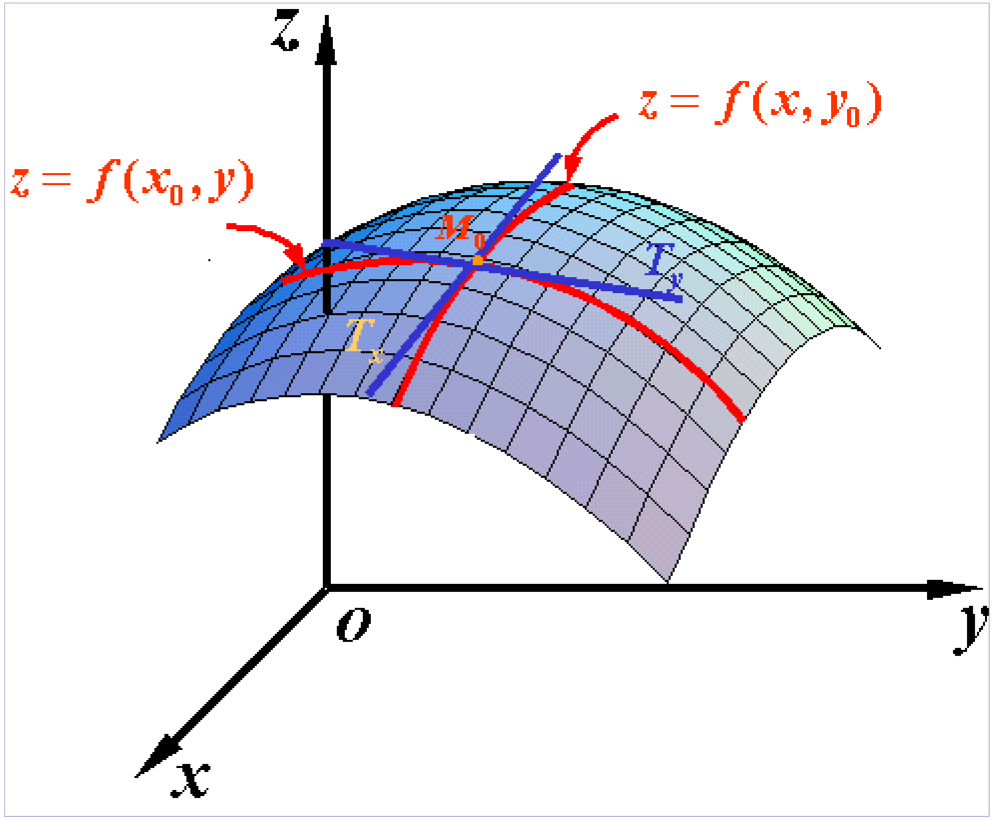
依定义知在 $(0,0)$ 处，$f_x(0,0)=f_y(0,0)=0.$

但函数在该点处并不连续.

偏导数的几何意义

设 $M_0(x_0, y_0, f(x_0, y_0))$ 为曲面 $z = f(x, y)$ 上一点，

如图



几何意义:

偏导数 $f_x(x_0,y_0)$就是曲面被平面 $y=y_0$ 所截得的曲线在点M_0处的切线M_0T_x对x轴的斜率.

偏导数 $f_y(x_0,y_0)$就是曲面被平面$x=x_0$ 所截得的曲线在点M_0处的切线M_0T_y对y轴的斜率.

二、高阶偏导数

函数$z = f(x,y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x,y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x,y)$$

纯偏导

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}(x,y), \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{yx}(x,y)$$

混合偏导

定义：二阶及二阶以上的偏导数统称为高阶偏导数.

例 5 设$z = x^3y^2 - 3xy^3 - xy + 1$，

求$\frac{\partial^2 z}{\partial x^2}$、$\frac{\partial^2 z}{\partial y \partial x}$、$\frac{\partial^2 z}{\partial x \partial y}$、$\frac{\partial^2 z}{\partial y^2}$及$\frac{\partial^3 z}{\partial x^3}$.

解 $\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$, $\frac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$;

$\frac{\partial^2 z}{\partial x^2} = 6xy^2$, $\frac{\partial^3 z}{\partial x^3} = 6y^2$, $\frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy$;

$\frac{\partial^2 z}{\partial x \partial y} = 6x^2y - 9y^2 - 1$, $\frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 - 1$.

例 6　设$u = e^{ax}\cos by$，求二阶偏导数.

解　$\dfrac{\partial u}{\partial x} = ae^{ax}\cos by$，　$\dfrac{\partial u}{\partial y} = -be^{ax}\sin by$；

$$\frac{\partial^2 u}{\partial x^2} = a^2 e^{ax}\cos by, \qquad \frac{\partial^2 u}{\partial y^2} = -b^2 e^{ax}\cos by,$$

$$\frac{\partial^2 u}{\partial x \partial y} = -abe^{ax}\sin by, \qquad \frac{\partial^2 u}{\partial y \partial x} = -abe^{ax}\sin by.$$

问题：混合偏导数都相等吗？具备怎样的条件才相等？

定理 如果函数$z = f(x, y)$的两个二阶混合偏导数 $\frac{\partial^2 z}{\partial y \partial x}$ 及 $\frac{\partial^2 z}{\partial x \partial y}$ 在区域 D 内连续，那末在该区域内这两个二阶混合偏导数必相等.

例 7 验证函数$u(x, y) = \ln\sqrt{x^2 + y^2}$满足拉普拉斯方程 $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

解 $\Theta \ln\sqrt{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2),$

$$\therefore \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}, \qquad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

$$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0.$$

例 8 设函数

$$f(x,y)=\begin{cases}\dfrac{x^3y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

求 $f_{xy}(0,0)$和$f_{yx}(0,0)$.

解 当$(x,y)\neq(0,0)$时，有

$$f_x(x,y)=\frac{3x^2y(x^2+y^2)-x^3y\cdot 2x}{(x^2+y^2)^2}$$

$$=\frac{3x^2y}{x^2+y^2}-\frac{2x^4y}{(x^2+y^2)^2}$$

$$f_y(x,y)=\frac{x^3}{x^2+y^2}-\frac{2x^3y^2}{(x^2+y^2)^2}$$

当$(x,y)=(0,0)$时，按定义有

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=0$$

$$f_{xy}(0,0)=\lim_{\Delta y\to 0}\frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y}=0$$

$$f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x}=1$$

作　　业

- P271 1(1，3，6); 2; 5(2); 7
- P271 6(选做)