§ 5.3 定积分的换元法和分部积分法

一、定积分的换元法

二、定积分的分部积分法

一、定积分的换元法

❖定理

假设函数$f(x)$在区间$[a, b]$上连续, 函数$x=\varphi(t)$满足条件:

(1)$\varphi(t)$在$[\alpha, \beta]$(或$[\beta, \alpha]$)上具有连续导数;

(2)$\varphi(\alpha)=a$, $\varphi(\beta)=b$, 且$\varphi([\alpha,\beta])=[a,b]$,

则有

$$\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt .$$

——换元公式

证明　设$F(x)$为$f(x)$的一个原函数,

则$F[\varphi(t)]$是$f[\varphi(t)]\varphi'(t)$的原函数，因此有

$$\int_a^b f(x)\,\mathrm{d}x = F(b)-F(a) = F[\varphi(\beta)]-F[\varphi(\alpha)]$$

$$= \int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t$$

例1 计算 $\int_0^a \sqrt{a^2 - x^2}\,\mathrm{d}x \ (a>0)$.

解 令 $x = a\sin t$, 则 $\mathrm{d}x = a\cos t\,\mathrm{d}t$,

当 $x=0$ 时, $t=0$; $x=a$ 时, $t=\frac{\pi}{2}$.

$$原式= a^2\int_0^{\frac{\pi}{2}} \cos^2 t\,\mathrm{d}t$$

$$= \frac{a^2}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2t)\,\mathrm{d}t$$

$$= \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)\Big|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4}$$

例2 计算 $\int_0^1 \frac{x}{\sqrt{4-3x}}\mathrm{d}x$.

解 令 $t=\sqrt{4-3x}$, 则 $x=\frac{4-t^2}{3}$, $\mathrm{d}x=-\frac{2}{3}t\mathrm{d}t$,

当$x=0$时, $t=2$; 当$x=1$时, $t=1$

$$原式=\int_2^1 \frac{\frac{4-t^2}{3}}{t}\cdot\left(-\frac{2}{3}t\right)\mathrm{d}t$$

$$=\frac{2}{9}\int_1^2 (4-t^2)\mathrm{d}t$$

$$=\left[\frac{2}{9}\left(4t-\frac{t^3}{3}\right)\right]_1^2=\frac{10}{27}.$$

$\int_a^b f(x)dx \xlongequal{令x=\varphi(t)} \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$ (当 $x=a$ 时 $t=\alpha$, 当 $x=b$ 时 $t=\beta$).

例3 计算 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx$.

解 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx = -\int_0^{\frac{\pi}{2}} \cos^5 x d\cos x$

$\xlongequal{令\cos x=t} -\int_1^0 t^5 dt = \int_0^1 t^5 dt = [\frac{1}{6}t^6]_0^1 = \frac{1}{6}$.

或 $\int_0^{\frac{\pi}{2}} \cos^5 x \sin x dx = -\int_0^{\frac{\pi}{2}} \cos^5 x d\cos x$

$= -[\frac{1}{6}\cos^6 x]_0^{\frac{\pi}{2}} = -\frac{1}{6}\cos^6\frac{\pi}{2} + \frac{1}{6}\cos^6 0 = \frac{1}{6}$.

提示：

换元一定要换积分限, 不换元积分限不变.

例4 设$f(x)$在$[-a, a]$上连续, 证明

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a}[f(x)+f(-x)]dx$$

并计算$\int_{-1}^{1}\frac{x^2}{1+e^x}dx.$

证明 因为$\int_{-a}^{a} f(x)dx=\int_{-a}^{0} f(x)dx+\int_{0}^{a} f(x)dx$,

而 $$\int_{-a}^{0} f(x)dx \overset{令x=-t}{=\!=\!=} -\int_{a}^{0} f(-t)dt = \int_{0}^{a} f(-t)dt = \int_{0}^{a} f(-x)dx,$$

$$\int_{-a}^{a} f(x)dx=\int_{0}^{a} f(-x)dx+\int_{0}^{a} f(x)dx=\int_{0}^{a}[f(-x)+f(x)]dx$$

$$\int_{-1}^{1}\frac{x^2}{1+e^x}dx = \int_{0}^{1}[\frac{x^2}{1+e^x}+\frac{(-x)^2}{1+e^{-x}}]dx = \int_{0}^{1}x^2dx = \frac{1}{3}.$$

例4 设$f(x)$在$[-a, a]$上连续, 证明

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [f(x) + f(-x)]dx$$

注: (1) 当$f(x)$为奇函数时, $\int_{-a}^{a} f(x)dx = 0.$

(2) 当$f(x)$为偶函数时, $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

练习 $\int_{-1}^{1} (\sin x + \sqrt{1+\cos^2 x})^2 dx =$ __4__.

$$(\sin x + \sqrt{1+\cos^2 x})^2$$

$$= \sin^2 x + 2\sin x \cdot \sqrt{1+\cos^2 x} + 1 + \cos^2 x$$

$$= 2 + 2\sin x \cdot \sqrt{1+\cos^2 x}$$

例5 若$f(x)$在[0, 1]上连续, 证明

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$;

(2) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

证明 (1)令 $x = \frac{\pi}{2} - t$, 则

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = -\int_{\frac{\pi}{2}}^{0} f[\sin(\frac{\pi}{2} - t)]dt$$

$$= \int_0^{\frac{\pi}{2}} f[\sin(\frac{\pi}{2} - t)]dt = \int_0^{\frac{\pi}{2}} f(\cos x)dx .$$

例5 若$f(x)$在[0, 1]上连续, 证明

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$;

(2) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

证明 (2)令$x=\pi-t$. 因为

$$\int_0^{\pi} xf(\sin x)dx = -\int_{\pi}^{0}(\pi-t)f[\sin(\pi-t)]dt$$

$$=\int_0^{\pi}(\pi-t)f[\sin(\pi-t)]dt = \int_0^{\pi}(\pi-t)f(\sin t)dt$$

$$=\pi\int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt$$

$$=\pi\int_0^{\pi} f(\sin x)dx - \int_0^{\pi} xf(\sin x)dx$$

所以 $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

例5 若$f(x)$在[0, 1]上连续, 证明

(1) $\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$;

(2) $\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

例6 计算 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x$

解 $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\mathrm{d}x$

$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}\mathrm{d}(\cos x)$

$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi} = -\frac{\pi}{2}(-\frac{\pi}{4}-\frac{\pi}{4}) = \frac{\pi^2}{4}.$

二、定积分的分部积分法

设函数$u(x)$、$v(x)$在区间$[a, b]$上具有连续导数.

由 $(uv)'=u'v+uv'$,

得 $uv'=(uv)'-u'v$,

等式两端在区间$[a, b]$上积分得

$$\int_a^b uv'dx=[uv]_a^b-\int_a^b u'vdx,$$

$$\int_a^b udv = uv\Big|_a^b - \int_a^b vdu$$

这就是定积分的分部积分公式.

例7 计算 $\int_0^{\frac{1}{2}} \arcsin x dx$.

解 $\int_0^{\frac{1}{2}} \arcsin x dx$

$$=[x\arcsin x]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} xd\arcsin x$$

$$=\frac{1}{2}\cdot\frac{\pi}{6} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}dx$$

$$=\frac{\pi}{12} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}d(1-x^2)$$

$$=\frac{\pi}{12} + [\sqrt{1-x^2}]_0^{\frac{1}{2}}$$

$$=\frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

例8 计算 $\int_0^1 \frac{\ln(1+x)}{(2-x)^2}\,\mathrm{d}x$

解 原式 $=\int_0^1 \ln(1+x)\,\mathrm{d}\left(\frac{1}{2-x}\right)$

$$=\frac{\ln(1+x)}{2-x}\Bigg|_0^1 - \int_0^1 \frac{1}{2-x}\cdot\frac{1}{1+x}\,\mathrm{d}x$$

$$=\ln 2 - \frac{1}{3}\int_0^1\left(\frac{1}{1+x}+\frac{1}{2-x}\right)\mathrm{d}x$$

$$=\ln 2 - \frac{1}{3}\big[\ln(1+x)-\ln(2-x)\big]_0^1$$

$$=\frac{1}{3}\ln 2$$

例9 求 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$.

解 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx = -\int_0^{\frac{\pi}{2}} \sin^{n-1} x d\cos x$

$= -[\cos x \sin^{n-1} x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x d\sin^{n-1} x$

$= (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x dx$

$= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2} x - \sin^n x) dx$

$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x dx$

$= (n-1)I_{n-2} - (n-1)I_n,$

由此得 $I_n = \dfrac{n-1}{n} I_{n-2}$.

例9 求 $I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx$.

公式: $I_n=\frac{n-1}{n}I_{n-2}$.

注意: $I_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}$, $I_1=\int_0^{\frac{\pi}{2}}\sin x dx=1$,

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx=\int_0^{\frac{\pi}{2}}\cos^n x dx$$

$$=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & n\text{为正偶数}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\cdots\frac{4}{5}\cdot\frac{2}{3}, & n\text{为大于1的正奇数}\end{cases}$$

例10 求 $\int_0^2 x^2\sqrt{4-x^2}\,dx$

解 令 $x=2\sin t$, $dx=2\cos t\,dt$

$$原式=\int_0^{\frac{\pi}{2}} 4\sin^2 t\cdot 2\cos t\cdot 2\cos t\,dt$$

$$=16\int_0^{\frac{\pi}{2}}(\sin^2 t-\sin^4 t)\,dt$$

$$=16\left(\frac{1}{2}\cdot\frac{\pi}{2}-\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)$$

$$=\pi.$$

思考题1

设 $f''(x)$ 在 $[0,1]$ 连续，且 $f(0)=1, f(2)=3,$

$f'(2)=5,$ 求 $\int_0^1 x f''(2x)\,\mathrm{d}x$.

解

$$\int_0^1 x f''(2x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 x\,\mathrm{d}f'(2x)$$

$$= \frac{1}{2}\left[\, x f'(2x)\Big|_0^1 - \int_0^1 f'(2x)\,\mathrm{d}x \,\right]$$

$$= \frac{5}{2} - \frac{1}{4} f(2x)\Big|_0^1$$

$$= 2$$

思考题2

设 $f(x)$ 在 $[a,b]$ 上有连续的二阶导数，

且 $f(a)=f(b)=0$，**试证**

$$\int_a^b f(x)\,\mathrm{d}x=\frac{1}{2}\int_a^b (x-a)(x-b)f''(x)\,\mathrm{d}x$$

证 右端 $=\dfrac{1}{2}\displaystyle\int_a^b (x-a)(x-b)\,\mathrm{d}f'(x)$

$$=\frac{1}{2}\Big[(x-a)(x-b)f'(x)\Big]\Big|_a^b-\frac{1}{2}\int_a^b f'(x)(2x-a-b)\,\mathrm{d}x$$

$$=-\frac{1}{2}\int_a^b (2x-a-b)\,\mathrm{d}f(x)$$

$$=-\frac{1}{2}\Big[(2x-a-b)f(x)\Big]\Big|_a^b+\int_a^b f(x)\,\mathrm{d}x=\text{左端}$$

作 业

习题5–3 (P249):

1.(4) (8) (12) (16) (20)

5.

8.

11.(3) (6) (9) (10) (12)