§4.2 换元积分法

一、第一类换元法

二、第二类换元法

微分运算中有两个重要法则:

复合函数微分法和乘积的微分法.

在积分运算中, 与它们对应的是本节的换元积分法和下节的分部积分法 ——

基本积分法 (两种).

一、第一类换元法

$$\int \cos 2x dx \neq \sin 2x + C \ ?$$

$$\int \cos x dx = \sin x + C$$

$$(\sin 2x)' = 2\cos 2x \neq \cos 2x$$

解决方法 将积分变量换成 $2x$. 因为 $dx = \frac{1}{2}d(2x)$

令 $t = 2x \Rightarrow dx = \frac{1}{2}dt$,

$$\int \cos 2x \, dx = \frac{1}{2}\int \cos t \, dt$$

$$= \frac{1}{2}\sin t + C = \frac{1}{2}\sin 2x + C$$

一、第一类换元法

❖定理1 (换元积分公式)

设 F 是 f 的一个原函数, $u=\varphi(x)$ 可导, 则有

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

$$[F(\varphi(x))]'=F'(\varphi(x))\cdot\varphi'(x)$$

$$=f(\varphi(x))\cdot\varphi'(x),$$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C$$

$$=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

一、第一类换元法

❖定理1 (换元积分公式)

设 F 是 f 的一个原函数, $u=\varphi(x)$ 可导, 则有

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

❖换元积分过程

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

凑微分: $\varphi'(x)\mathrm{d}x=\mathrm{d}\varphi(x)$

$$=\int f[\varphi(x)]\mathrm{d}\varphi(x)$$

换元: $\int f(x)\mathrm{d}x=F(x)+C$

$\rightarrow \int f[\varphi(x)]\mathrm{d}\varphi(x)=F[\varphi(x)]+C$

$$=F[\varphi(x)]+C$$

关键点: 如何确定中间变量 $u=\varphi(x)$?

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f[\varphi(x)]\mathrm{d}\varphi(x), \quad u = \varphi(x) = ?$$

从被积函数中**明显的复合部分**去确定 u

例1 $\int 2\cos 2x dx$

$u = 2x,$

$u' = 2.$

$= \int \cos 2x \cdot (2x)' dx = \int \cos 2x d(2x)$

$= \int \cos u du = \sin u + C = \sin 2x + C .$

例2 $\int 2x e^{x^2} dx$

$u = x^2,$

$u' = 2x.$

$= \int e^{x^2} (x^2)' dx = \int e^{x^2} d(x^2)$

$= \int e^u du = e^u + C = e^{x^2} + C .$

从被积函数中**明显的复合部分**去确定 u

例3 $\int x\sqrt{1-x^2}dx$

$u = 1 - x^2$,

$u' = -2x$.

$$= -\frac{1}{2}\int\sqrt{1-x^2}(1-x^2)'dx$$

$$= -\frac{1}{2}\int\sqrt{1-x^2}d(1-x^2)$$

$$= -\frac{1}{2}\int u^{\frac{1}{2}}du = -\frac{1}{3}u^{\frac{3}{2}} + C$$

$$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + C .$$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f[\varphi(x)]\mathrm{d}\varphi(x), \quad u = \varphi(x) = ?$$

从被积函数中明显的复合部分去确定 u

例4 $\displaystyle\int \frac{e^{3\sqrt{x}}}{\sqrt{x}}\mathrm{d}x = \frac{2}{3}\int e^{3\sqrt{x}}(3\sqrt{x})'\mathrm{d}x$

$$= \frac{2}{3}\int e^{3\sqrt{x}}\mathrm{d}(3\sqrt{x}) = \frac{2}{3}e^{3\sqrt{x}} + C.$$

$u = 3\sqrt{x},$

$u' = \dfrac{3}{2\sqrt{x}}.$

例5 $\displaystyle\int \frac{1}{x^2}(1-\frac{1}{x})^9\mathrm{d}x$

$$= \int (1-\frac{1}{x})^9(1-\frac{1}{x})'\mathrm{d}x$$

$$= \int (1-\frac{1}{x})^9\mathrm{d}(1-\frac{1}{x}) = \frac{1}{10}(1-\frac{1}{x})^{10} + C.$$

$u = 1-\dfrac{1}{x},$

$u' = \dfrac{1}{x^2}.$

从被积函数中**明显的复合部分**去确定 u

例6 求 $\int \frac{1+\ln x}{(x\ln x)^2}\mathrm{d}x.$

解

$$\int \frac{1+\ln x}{(x\ln x)^2}\mathrm{d}x$$

$u = x\ln x,$

$\mathrm{d}u = (1+\ln x)\mathrm{d}x.$

$$= \int \frac{1}{(x\ln x)^2}\mathrm{d}(x\ln x)$$

$$= -\frac{1}{x\ln x} + C.$$

从被积函数中**明显的复合部分**去确定 u

例7 求 $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\mathrm{d}x.$

$u = \tan x,$

解 $\int \frac{\sqrt{\tan x}}{\sin x \cos x}\mathrm{d}x$

$\mathrm{d}(\tan x) = \sec^2 x\mathrm{d}x,$

$= \int \frac{\sqrt{\tan x}}{\tan x \cos^2 x}\mathrm{d}x$

$\mathrm{d}(\tan x) = \frac{1}{\cos^2 x}\mathrm{d}x.$

$= \int \frac{1}{\sqrt{\tan x}}\mathrm{d}(\tan x)$

$= 2\sqrt{\tan x} + C.$

通过凑微分确定 u

例8 $\int \frac{\ln x}{x}\mathrm{d}x$

$$\frac{1}{x}\mathrm{d}x = \mathrm{d}(\ln x)$$

$$= \int \ln x\,\mathrm{d}(\ln x)$$

$$= \frac{1}{2}\ln^2 x + C.$$

例9 $\int \frac{x}{1+x^4}\mathrm{d}x$

$$x\mathrm{d}x = \frac{1}{2}\mathrm{d}(x^2)$$

$$= \frac{1}{2}\int \frac{1}{1+(x^2)^2}\mathrm{d}(x^2)$$

$$= \frac{1}{2}\arctan(x^2) + C.$$

通过凑微分确定 u

例10 $\displaystyle \int \frac{e^x}{e^x+1}\mathrm{d}x = \int \frac{1}{e^x+1}\mathrm{d}(e^x+1)$

$e^x\mathrm{d}x = \mathrm{d}(e^x)$

$$= \ln(e^x+1)+C.$$

例11 $\displaystyle \int \frac{1}{e^x+e^{-x}}\mathrm{d}x = \int \frac{e^x}{e^{2x}+1}\mathrm{d}x$

$$= \int \frac{1}{(e^x)^2+1}\mathrm{d}(e^x)$$

$$= \arctan(e^x)+C.$$

例12 求 $\int \frac{1}{1+e^x}\mathrm{d}x$

解 法一 $\int \frac{1}{1+e^x}\mathrm{d}x = \int \frac{1+e^x-e^x}{1+e^x}\mathrm{d}x$

$e^x\mathrm{d}x = \mathrm{d}(e^x)$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)\mathrm{d}x = \int \mathrm{d}x - \int \frac{e^x}{1+e^x}\mathrm{d}x$$

$$= \int \mathrm{d}x - \int \frac{1}{1+e^x}\mathrm{d}(1+e^x)$$

$$= x - \ln(1+e^x) + C$$

例12　求 $\int \frac{1}{1+e^x}\mathrm{d}x$

法二　$\int \frac{1}{1+e^x}\mathrm{d}x = \int \frac{e^x}{e^x(1+e^x)}\mathrm{d}x$

$e^x\mathrm{d}x = \mathrm{d}(e^x)$

$$= \int \frac{1}{e^x(1+e^x)}\mathrm{d}e^x = \int \frac{1}{u(1+u)}\mathrm{d}u$$

$u = e^x$

$$= \int \frac{(u+1)-u}{u(1+u)}\mathrm{d}u = \int \left(\frac{1}{u} - \frac{1}{u+1}\right)\mathrm{d}u$$

$$= \int \frac{1}{u}\mathrm{d}u - \int \frac{1}{u+1}\mathrm{d}(u+1)$$

$$= \ln u - \ln(u+1) + C = \ln \frac{e^x}{e^x+1} + C$$

例12 求 $\int \frac{1}{1+e^x}\mathrm{d}x$

法三 $\int \frac{1}{1+e^x}\mathrm{d}x$

$e^{-x}\mathrm{d}x = -\mathrm{d}(e^{-x})$

$$= \int \frac{e^{-x}}{e^{-x}+1}\mathrm{d}x$$

$$= -\int \frac{1}{e^{-x}+1}\mathrm{d}(e^{-x})$$

$$= -\int \frac{1}{e^{-x}+1}\mathrm{d}(e^{-x}+1)$$

$$= -\ln(e^{-x}+1)+C.$$

例13 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2}dx = \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}d\frac{x}{a}$

$= \frac{1}{a}\arctan\frac{x}{a} + C$.

例14 当$a>0$时,

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \frac{1}{a}\int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}dx = \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}d\frac{x}{a} = \arcsin\frac{x}{a} + C .$$

积分公式:

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a} + C , \quad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C .$$

例15 $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\int (\frac{1}{x-a} - \frac{1}{x+a})dx$

$= \frac{1}{2a}[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx]$

$= \frac{1}{2a}[\int \frac{1}{x-a}d(x-a) - \int \frac{1}{x+a}d(x+a)]$

$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + C$

$= \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C$.

积分公式： $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln|\frac{x-a}{x+a}| + C$.

一些三角函数的积分

例16 求$\int \tan x dx$.

解 原式$=\int \frac{\sin x}{\cos x} dx$

$$= -\int \frac{d\cos x}{\cos x}$$

$$= -\ln|\cos x| + C$$

积分公式:

$$\int \tan x dx = -\ln|\cos x| + C, \quad \int \cot x dx = \ln|\sin x| + C.$$

一些三角函数的积分

例17 求 $\int \csc x \mathrm{d}x.$

解 法一 $\int \csc x \mathrm{d}x = \int \frac{1}{\sin x} \mathrm{d}x = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} \mathrm{d}x$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} \mathrm{d}\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} \mathrm{d}\left(\tan\frac{x}{2}\right)$$

$$= \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C$$

积分公式: $\int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C$.

一些三角函数的积分

例17 求 $\int \csc x \mathrm{d}x.$

法二 $\int \csc x \mathrm{d}x = \int \frac{1}{\sin x}\mathrm{d}x = \int \frac{\sin x}{\sin^2 x}\mathrm{d}x$

$= -\int \frac{1}{1-\cos^2 x}\mathrm{d}(\cos x)$ $\quad u = \cos x$

$= -\int \frac{1}{1-u^2}\mathrm{d}u$ $\quad \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$

$= \frac{1}{2}\ln\frac{1-u}{1+u} + C$

$= \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C$

一些三角函数的积分

例18 $\int \sec x dx$

$$= \int \csc(x+\frac{\pi}{2})dx = \int \csc(x+\frac{\pi}{2})d(x+\frac{\pi}{2})$$

$$= \ln|\csc(x+\frac{\pi}{2}) - \cot(x+\frac{\pi}{2})| + C$$

$$= \ln|\sec x + \tan x| + C.$$

积分公式:

$$\int \sec x dx = \ln|\sec x + \tan x| + C\ .$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C\ .$$

一些三角函数的积分

例19 求 $\int \sin^2 x \cdot \cos^5 x \mathrm{d}x.$

$\cos x\mathrm{d}x = \mathrm{d}(\sin x)$

解 $\int \sin^2 x \cdot \cos^5 x \mathrm{d}x$

$$= \int \sin^2 x \cdot \cos^4 x \mathrm{d}(\sin x)$$

$$= \int \sin^2 x \cdot (1 - \sin^2 x)^2 \mathrm{d}(\sin x)$$

$$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) \mathrm{d}(\sin x)$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$$

一些三角函数的积分

例20 求 $\int \sin^2 x \mathrm{d}x.$

解 $\int \sin^2 x \mathrm{d}x = \frac{1}{2}\int (1-\cos 2x)\mathrm{d}x$

$$= \frac{1}{2}\int \mathrm{d}x - \frac{1}{4}\int \cos 2x \mathrm{d}2x = \frac{x}{2} - \frac{1}{4}\sin 2x + C$$

$\sin^m x \cos^n x$积分

$$\sin^m x \cdot \cos^{2k+1} x = \sin^m x \cdot (1-\sin^2 x)^k (\sin x)'$$

$$\cos^n x \cdot \sin^{2k+1} x = -\cos^n x \cdot (1-\cos^2 x)^k (\cos x)'$$

当m、n都是偶数时，用倍角公式降幂.

一些三角函数的积分

例21 求 $\int \frac{1}{1+\cos x}\mathrm{d}x.$

解 法一

$$\int \frac{1}{1+\cos x}\mathrm{d}x = \int \frac{1}{2\cos^2 \frac{x}{2}}\mathrm{d}x$$

$$= \int \frac{1}{\cos^2 \frac{x}{2}}\mathrm{d}\left(\frac{x}{2}\right)$$

$$= \tan \frac{x}{2} + C$$

一些三角函数的积分

例21 求 $\int \frac{1}{1+\cos x}\mathrm{d}x.$

解 法二

$$\int \frac{1}{1+\cos x}\mathrm{d}x = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\mathrm{d}x$$

$$= \int \frac{1-\cos x}{1-\cos^2 x}\mathrm{d}x = \int \frac{1-\cos x}{\sin^2 x}\mathrm{d}x$$

$$= \int \frac{1}{\sin^2 x}\mathrm{d}x - \int \frac{1}{\sin^2 x}\mathrm{d}(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C$$

一些三角函数的积分

例22 求 $\int \frac{\cos x}{\sin x+\cos x}\mathrm{d}x.$

解 $\int \frac{\cos x}{\sin x+\cos x}\mathrm{d}x$

$$\frac{\cos x-\sin x}{\sin x+\cos x}=\frac{(\sin x+\cos x)'}{\sin x+\cos x}$$

$$=\frac{1}{2}\int \frac{\cos x+\sin x+(\cos x-\sin x)}{\sin x+\cos x}\mathrm{d}x$$

$$=\frac{x}{2}+\frac{1}{2}\int \frac{1}{\sin x+\cos x}\mathrm{d}(\sin x+\cos x)$$

$$=\frac{x}{2}+\frac{1}{2}\ln|\sin x+\cos x|+C$$

一些三角函数的积分

例23　求 $\int \tan^3 x \sec^4 x dx.$

解　法一

$$\int \tan^3 x \sec^4 x dx$$

$$\sec^2 x dx = d(\tan x)$$

$$= \int \tan^3 x \sec^2 x d(\tan x)$$

$$= \int \tan^3 x(\tan^2 x + 1) d(\tan x)$$

$$= \int (\tan^5 x + \tan^3 x) d(\tan x)$$

$$= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C.$$

一些三角函数的积分

例23 求 $\int \tan^3 x \sec^4 x dx.$

解 法二

$$\int \tan^3 x \sec^4 x dx$$

$$\sec x \tan x dx = d(\sec x)$$

$$= \int \tan^2 x \sec^3 x d(\sec x)$$

$$= \int (\sec^2 x - 1) \sec^3 x d(\sec x)$$

$$= \int (\sec^5 x - \sec^3 x) d(\sec x)$$

$$= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C.$$

常见的凑微分类型

$$\int f(ax+b)\mathrm{d}x = \frac{1}{a}\int f(ax+b)\mathrm{d}(ax+b)\ (a \neq 0)$$

$$\int f(x^{m+1})x^m \mathrm{d}x = \frac{1}{m+1}\int f(x^{m+1})\mathrm{d}(x^{m+1})$$

$$\int f(\sqrt{x})\frac{\mathrm{d}x}{\sqrt{x}} = 2\int f(\sqrt{x})\mathrm{d}(\sqrt{x})$$

$$\int f(\frac{1}{x})\frac{\mathrm{d}x}{x^2} = -\int f(\frac{1}{x})\mathrm{d}(\frac{1}{x})$$

$$\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\mathrm{d}(\ln x)$$

$$\int f(e^x)e^x \mathrm{d}x = \int f(e^x)\mathrm{d}(e^x)$$

常见的凑微分类型

$$\int f(\sin x)\cos x\mathrm{d}x=\int f(\sin x)\mathrm{d}\sin x$$

$$\int f(\cos x)\sin x\mathrm{d}x=-\int f(\cos x)\mathrm{d}\cos x$$

$$\int f(\tan x)\sec^2 x\mathrm{d}x=\int f(\tan x)\mathrm{d}\tan x$$

$$\int f(\sec x)\sec x\tan x\,\mathrm{d}x=\int f(\sec x)\mathrm{d}\sec x$$

$$\int f(\arcsin x)\frac{1}{\sqrt{1-x^2}}\mathrm{d}x=\int f(\arcsin x)\mathrm{d}\arcsin x$$

$$\int f(\arctan x)\frac{1}{1+x^2}\mathrm{d}x=\int f(\arctan x)\mathrm{d}\arctan x$$

$$\int\frac{f'(x)}{f(x)}\mathrm{d}x=\int\frac{1}{f(x)}\mathrm{d}f(x)=\ln|f(x)|+C$$

作 业

习题4–2 (P204):

2. (3) (6) (9) (12) (15) (18)
(21) (23) (24) (30) (31) (33)