



# 第二章 导数与微分

## 习题课

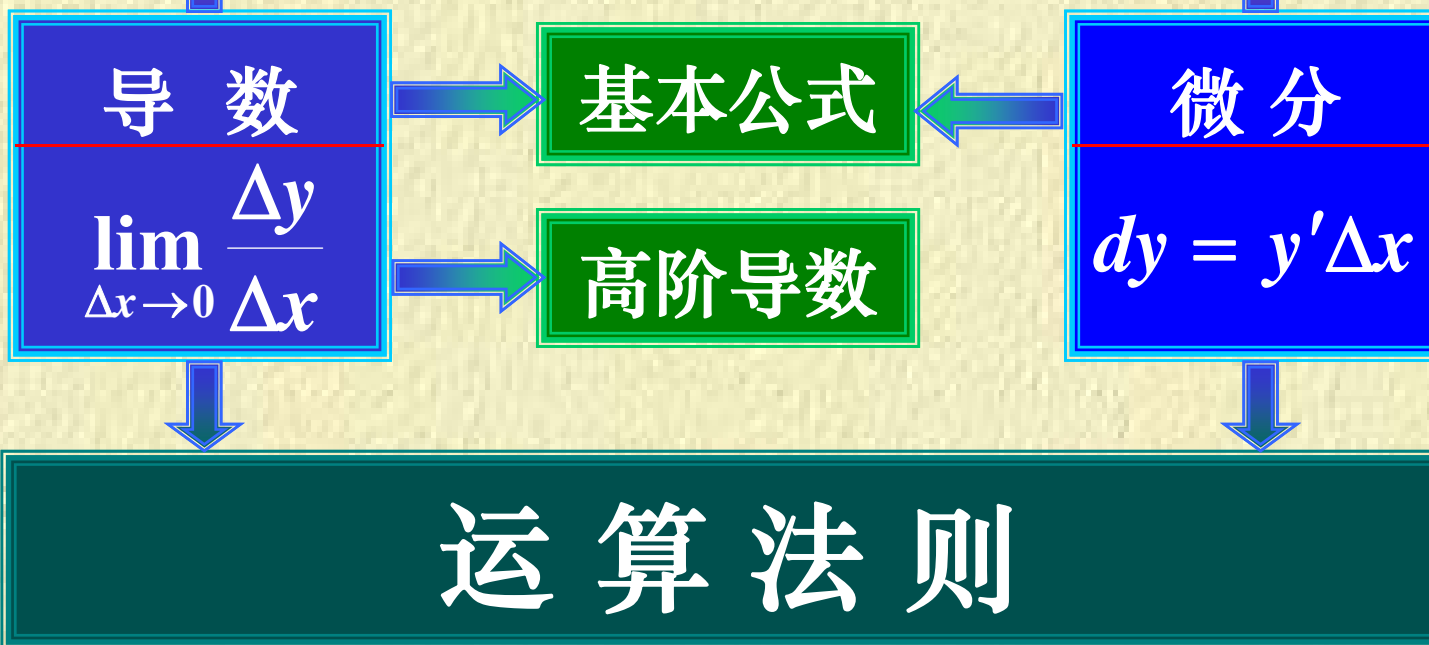
一、主要内容框图

二、典型例题



# 一、主要内容框图

关系  $\frac{dy}{dx} = y' \Leftrightarrow dy = y'dx \Leftrightarrow \Delta y = dy + o(\Delta x)$





## 二、典型例题

**例1** 设  $f(x) = x(x-1)(x-2)\Lambda(x-100)$ ,  
求  $f'(0)$ .

**解1** 
$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{x \rightarrow 0} (x-1)(x-2)\Lambda(x-100) = 100!$$

**解2** 
$$f'(x) = (x-1)\Lambda(x-100) + x[(x-1)\Lambda(x-100)]'$$
$$f'(0) = (-1)(-2)\Lambda(-100) = 100!$$



**例2** 设  $f(x)$  在  $x = 2$  处连续, 且  $\lim_{x \rightarrow 2} \frac{f(x)}{x-2} = 3$ ,  
求  $f'(2)$ .

**解**  $f(2) = \lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} [(x-2) \cdot \frac{f(x)}{(x-2)}] = 0$

$$f'(2) = \lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{f(x)}{x - 2} = 3$$



**例3** 若  $f(1) = 0$  且  $f'(1)$  存在, 求  $\lim_{x \rightarrow 0} \frac{f(\sin^2 x + \cos x)}{(e^x - 1) \tan x}$ .

**解** 原式 =  $\lim_{x \rightarrow 0} \frac{f(\sin^2 x + \cos x)}{x^2}$

$$\lim_{x \rightarrow 0} (\sin^2 x + \cos x) = 1 \text{ 且 } f(1) = 0$$

联想到凑导数的定义式

$$= \lim_{x \rightarrow 0} \frac{f(1 + \sin^2 x + \cos x - 1) - f(1)}{\sin^2 x + \cos x - 1} \cdot \frac{\sin^2 x + \cos x - 1}{x^2}$$

$$= f'(1) \cdot \left(1 - \frac{1}{2}\right) = \frac{1}{2} f'(1)$$



**例4** 设  $f(x) = x|x(x-2)|$ , 求  $f'(x)$ .

**解** 先去掉绝对值

$$f(x) = \begin{cases} x^2(x-2), & x \leq 0 \\ -x^2(x-2), & 0 < x < 2, \\ x^2(x-2), & x \geq 2 \end{cases}$$

$$\begin{aligned} f'_-(0) &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0^-} \frac{x^2(x-2)}{x} = 0, \end{aligned}$$

$$\begin{aligned} f'_+(0) &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0^+} \frac{-x^2(x-2)}{x} = 0, \end{aligned}$$

$$f'_-(0) = f'_+(0) = 0, \quad f'(0) = 0;$$

$$\begin{aligned} f'_-(2) &= \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{-x^2(x-2)}{x-2} = -4, \end{aligned}$$

$$\begin{aligned} f'_+(2) &= \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2^+} \frac{x^2(x-2)}{x-2} = 4, \end{aligned}$$

$$f'_-(2) \neq f'_+(2),$$

$\therefore f(x)$  在  $x = 2$  处不可导.



**例4** 设  $f(x) = x|x(x-2)|$ , 求  $f'(x)$ .

**解** 先去掉绝对值

$$f(x) = \begin{cases} x^2(x-2), & x \leq 0 \\ -x^2(x-2), & 0 < x < 2, \\ x^2(x-2), & x \geq 2 \end{cases}$$

$$f'(x) = \begin{cases} 3x^2 - 4x, & x < 0 \\ 0, & x = 0 \\ -3x^2 + 4x, & 0 < x < 2 \\ 3x^2 - 4x, & x > 2 \end{cases}$$

$$f'_-(0) = f'_+(0) = 0, \quad f'(0) = 0;$$

$$\begin{aligned} f'_-(2) &= \lim_{x \rightarrow 2^-} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2^-} \frac{-x^2(x-2)}{x-2} = -4, \end{aligned}$$

$$\begin{aligned} f'_+(2) &= \lim_{x \rightarrow 2^+} \frac{f(x) - f(2)}{x - 2} \\ &= \lim_{x \rightarrow 2^+} \frac{x^2(x-2)}{x-2} = 4, \end{aligned}$$

$$f'_-(2) \neq f'_+(2),$$

$\therefore f(x)$  在  $x = 2$  处不可导.



**例5** 设  $y = e^{\sin x} \sin e^x + f(\arctan \frac{1}{x})$ , 其中  $f(x)$  可微, 求  $y'$ .

**解**

$$\begin{aligned} y' &= \sin e^x \cdot (e^{\sin x})' + e^{\sin x} (\sin e^x)' \\ &\quad + f'(\arctan \frac{1}{x}) (\arctan \frac{1}{x})' \\ &= \sin e^x \cdot e^{\sin x} (\sin x)' + e^{\sin x} \cdot \cos e^x \cdot (e^x)' \\ &\quad + f'(\arctan \frac{1}{x}) \cdot \frac{1}{1 + \frac{1}{x^2}} \cdot (\frac{1}{x})' \\ &= e^{\sin x} (\cos x \sin e^x + e^x \cos e^x) \\ &\quad - \frac{1}{x^2 + 1} f'(\arctan \frac{1}{x}) \end{aligned}$$





**例6** 设  $y = \sin^2 \ln(e^x + x)$ , 求  $dy$ .

**解**

$$\begin{aligned}y' &= 2 \sin \ln(e^x + x) \cdot [\sin \ln(e^x + x)]' \\&= 2 \sin \ln(e^x + x) \cdot [\cos \ln(e^x + x)] \cdot [\ln(e^x + x)]' \\&= \sin[2 \ln(e^x + x)] \cdot \frac{1}{e^x + x} \cdot (e^x + x)' \\&= \frac{e^x + 1}{e^x + x} \sin[2 \ln(e^x + x)], \\dy &= y' dx = \frac{e^x + 1}{e^x + x} \sin[2 \ln(e^x + x)] dx.\end{aligned}$$



**例7** 设函数  $y = f(x)$  由方程  $\sqrt[x]{y} = \sqrt[y]{x} (x > 0, y > 0)$

所确定, 求  $\frac{d^2 y}{dx^2}$ .

**解** 两边取对数  $\frac{1}{x} \ln y = \frac{1}{y} \ln x$ , 即  $y \ln y = x \ln x$ ,

$$\therefore (\ln y + 1)y' = \ln x + 1, \quad y' = \frac{\ln x + 1}{\ln y + 1},$$

$$y'' = \frac{\frac{1}{x}(\ln y + 1) - (\ln x + 1)\frac{1}{y} \cdot y'}{(\ln y + 1)^2}$$

$$= \frac{y(\ln y + 1)^2 - x(\ln x + 1)^2}{xy(\ln y + 1)^3}$$



**例8** 设  $y = x(\sin x)^{\cos x}$ , 求  $y'$ .

$$(\ln y)' = \frac{1}{y} \cdot y'$$

**解**  $y' = y(\ln y)'$

$$= y(\ln x + \cos x \ln \sin x)'$$

$$= x(\sin x)^{\cos x} \left( \frac{1}{x} - \sin x \cdot \ln \sin x + \frac{\cos^2 x}{\sin x} \right)$$



**例9** 
$$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}, \text{求 } \frac{d^2 y}{dx^2}.$$

**解** 
$$\frac{dy}{dx} = \frac{[a(1 - \cos t)]'}{[a(t - \sin t)]'} = \frac{a \sin t}{a(1 - \cos t)} = \frac{\sin t}{1 - \cos t},$$

$$\left(\frac{dy}{dx}\right)'_t = \frac{\cos t \cdot (1 - \cos t) - \sin t \cdot (\sin t)}{(1 - \cos t)^2} = \frac{1}{\cos t - 1},$$

$$\frac{d^2 y}{dx^2} = \frac{\left(\frac{dy}{dx}\right)'_t}{x'(t)} = -\frac{1}{a(1 - \cos t)^2}.$$



**例10** 设由方程 
$$\begin{cases} x = t^2 + 2t \\ t^2 - y + \varepsilon \sin y = 1 \quad (0 < \varepsilon < 1) \end{cases}$$

确定函数  $y = y(x)$ , 求  $\frac{dy}{dx}$ .

**解** 方程组两边对  $t$  求导, 得

$$\begin{cases} \frac{dx}{dt} = 2t + 2 \\ 2t - \frac{dy}{dt} + \varepsilon \cos y \frac{dy}{dt} = 0 \end{cases} \longrightarrow \begin{cases} \frac{dx}{dt} = 2(t + 1) \\ \frac{dy}{dt} = \frac{2t}{1 - \varepsilon \cos y} \end{cases}$$

故 
$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{t}{(t + 1)(1 - \varepsilon \cos y)}$$



**例11** 设  $y = \frac{4x^2 - 1}{x^2 - 1}$ , 求  $y^{(n)}$ .

**解** 
$$y = \frac{4x^2 - 1}{x^2 - 1} = \frac{4x^2 - 4 + 3}{x^2 - 1}$$

$$= 4 + \frac{3}{2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right)$$

$$\ominus \left( \frac{1}{x-1} \right)^{(n)} = \frac{(-1)^n n!}{(x-1)^{n+1}}, \quad \left( \frac{1}{x+1} \right)^{(n)} = \frac{(-1)^n n!}{(x+1)^{n+1}},$$

$$\therefore y^{(n)} = \frac{3}{2} (-1)^n n! \left[ \frac{1}{(x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}} \right].$$



**例12** 求曲线  $\begin{cases} x = \frac{3at}{1+t^2} \\ y = \frac{3at^2}{1+t^2} \end{cases}$  在  $t = 2$  相应的点处的切线方程

**解** 切点坐标为  $(\frac{6a}{5}, \frac{12a}{5})$ ,

$$x'(t) = \frac{3a(1+t^2) - 3at \cdot 2t}{(1+t^2)^2}$$

$$= \frac{3a(1-t^2)}{(1+t^2)^2},$$

$$y'(t) = \frac{6at(1+t^2) - 3at^2 \cdot 2t}{(1+t^2)^2}$$

$$= \frac{6at}{(1+t^2)^2},$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t}{1-t^2},$$

$$k = \left. \frac{dy}{dx} \right|_{t=2} = -\frac{4}{3},$$

切线方程为

$$y - \frac{12a}{5} = -\frac{4}{3} \left( x - \frac{6a}{5} \right),$$

即  $4x + 3y - 12a = 0.$



# 测试题

一、求下列函数的导数：

$$(1) y = \arctan \frac{1-x^2}{1+x^2};$$

$$(2) y = \cos^2(e^x + e^{-x});$$

$$(3) y = x\sqrt{x^2-1} - \ln(x + \sqrt{x^2-1});$$

$$(4) y = \sqrt{x^2-1} - \arccos \frac{1}{x} \quad (x > 1);$$

$$(5) y = \left( \frac{x}{1+x} \right)^x.$$





# 测试题

二、  $\begin{cases} x = t^3 + 3t \\ y = t^5 - 5t \end{cases}$ , 求  $\frac{d^2 y}{dt^2}$ ,  $\frac{d}{dt} \left( \frac{dy}{dx} \right)$ ,  $\frac{d^2 y}{dx^2}$ .

三、 求曲线  $\begin{cases} e^x = 3t^2 + 2t + 1 \\ t \sin y - y + \frac{\pi}{2} = 0 \end{cases}$  在  $t = 0$

相应的点处的切线方程.

四、  $f(x) = \begin{cases} e^x + x, & x \leq 0 \\ 2 \sin x + a, & x > 0 \end{cases}$ , 求  $f'(x)$ .



# 测试题

五、设  $f(x) = \begin{cases} x^2 \sin \frac{1}{x} + a, & x < 0 \\ e^x + b \sin x, & x \geq 0 \end{cases}$  在点  $x = 0$  可导,  
求  $a, b$  的值.

六、设  $f(x)$  在  $x = 1$  处连续, 且  $\lim_{x \rightarrow 1} \frac{f(x) + \ln x}{x^2 - 1} = 3$ ,  
求  $f'(1)$ .



# 测试题解答

一、求下列函数的导数：

$$(1) y = \arctan \frac{1-x^2}{1+x^2}; \quad (1) y' = -\frac{2x}{1+x^4};$$

$$(2) y = \cos^2(e^x + e^{-x}); \quad (2) y' = (e^{-x} - e^x) \sin 2(e^x + e^{-x});$$

$$(3) y = x\sqrt{x^2-1} - \ln(x + \sqrt{x^2-1}); \quad (3) y' = 2\sqrt{x^2-1};$$

$$(4) y = \sqrt{x^2-1} - \arccos \frac{1}{x} \quad (x > 1); \quad (4) y' = \frac{\sqrt{x^2-1}}{x};$$

$$(5) y = \left(\frac{x}{1+x}\right)^x. \quad (5) y' = \left(\frac{x}{1+x}\right)^x \left(\ln \frac{x}{1+x} + \frac{1}{1+x}\right).$$



二、 
$$\begin{cases} x = t^3 + 3t \\ y = t^5 - 5t \end{cases}, \text{求 } \frac{d^2 y}{dt^2}, \frac{d}{dt} \left( \frac{dy}{dx} \right), \frac{d^2 y}{dx^2}.$$

解 
$$\frac{dy}{dt} = 5t^4 - 5, \quad \frac{d^2 y}{dt^2} = 20t^3,$$

$$\frac{dx}{dt} = 3t^2 + 3, \quad \frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{5t^4 - 5}{3t^2 + 3} = \frac{5}{3}(t^2 - 1),$$

$$\frac{d}{dt} \left( \frac{dy}{dx} \right) = \left[ \frac{5}{3}(t^2 - 1) \right]' = \frac{10}{3}t,$$

$$\frac{d^2 y}{dx^2} = \frac{\left( \frac{dy}{dx} \right)'_t}{x'(t)} = \frac{10t}{9(t^2 + 1)}.$$



三、求曲线  $\begin{cases} e^x = 3t^2 + 2t + 1 \\ t \sin y - y + \frac{\pi}{2} = 0 \end{cases}$  在  $t = 0$

相应的点处的切线方程.

解  $e^x \frac{dx}{dt} = 6t + 2, \frac{dx}{dt} = \frac{6t + 2}{e^x},$

$$\sin y + t \cos y \cdot \frac{dy}{dt} - \frac{dy}{dt} = 0, \frac{dy}{dt} = \frac{\sin y}{1 - t \cos y},$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{e^x \sin y}{(6t + 2)(1 - t \cos y)},$$

$$t = 0, x = 0, y = \frac{\pi}{2}, k = \left. \frac{dy}{dx} \right|_{t=0, x=0, y=\frac{\pi}{2}} = \frac{1}{2},$$

所求切线方程为  $y - \frac{\pi}{2} = \frac{1}{2}x.$



四、  $f(x) = \begin{cases} e^x + x, & x \leq 0 \\ 2\sin x + a, & x > 0 \end{cases}$ , 求  $f'(x)$ .

解  $\lim_{x \rightarrow 0^-} f(x) = f(0) = 1,$

$\lim_{x \rightarrow 0^+} f(x) = a,$

1. 当  $a \neq 1$  时,  $f(x)$  在点  $x = 0$  不连续, 从而不可导.

$$f'(x) = \begin{cases} e^x + 1, & x < 0 \\ 2\cos x, & x > 0 \end{cases}$$

2. 当  $a = 1$  时,

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{e^x + x - 1}{x} = 2,$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{2\sin x + a - 1}{x} = 2,$$

• Q  $f'_-(0) = f'_+(0) = 2,$

$\therefore f'(0) = 2,$

$$f'(x) = \begin{cases} e^x + 1, & x \leq 0 \\ 2\cos x, & x > 0 \end{cases}.$$



五、设  $f(x) = \begin{cases} x^2 \sin \frac{1}{x} + a, & x < 0 \\ e^x + b \sin x, & x \geq 0 \end{cases}$  在点  $x = 0$  可导,

求  $a, b$  的值.

**解**  $\lim_{x \rightarrow 0^-} f(x) = a, \lim_{x \rightarrow 0^+} f(x) = f(0) = 1,$

因  $f(x)$  在点  $x = 0$  可导, 从而连续, 于是  $a = 1.$

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0,$$

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^+} \frac{e^x + b \sin x - 1}{x} = 1 + b,$$

由  $f(x)$  在点  $x = 0$  可导, 得  $b + 1 = 0, b = -1.$



六、 设  $f(x)$  在  $x=1$  处连续, 且  $\lim_{x \rightarrow 1} \frac{f(x) + \ln x}{x^2 - 1} = 3$ ,  
求  $f'(1)$ .

**解**  $\lim_{x \rightarrow 1} [f(x) + \ln x] = 0 \implies f(1) = 0$

$$\lim_{x \rightarrow 1} \frac{f(x) + \ln x}{x^2 - 1} = \lim_{x \rightarrow 1} \frac{f(x)}{x^2 - 1} + \lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$$

$$= \frac{1}{2} \lim_{x \rightarrow 1} \frac{f(x)}{x - 1} + \frac{1}{2} \lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \frac{1}{2} \lim_{x \rightarrow 1} \frac{f(x)}{x - 1} + \frac{1}{2} = 3$$

$$\implies \lim_{x \rightarrow 1} \frac{f(x)}{x - 1} = 5$$

$$f'(1) = \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \rightarrow 1} \frac{f(x)}{x - 1} = 5$$