



## § 2.4 高阶导数

- ❖ 高阶导数的定义
- ❖ 高阶导数求法举例



## 高阶导数的定义

**问题** 变速直线运动的加速度.

设  $s = f(t)$ , 则瞬时速度为  $v(t) = f'(t)$

⊕ 加速度  $\alpha$  是速度  $v$  对时间  $t$  的变化率

$$\therefore \alpha(t) = v'(t) = [f'(t)]'.$$

**定义** 如果函数  $f(x)$  的导数  $f'(x)$  在点  $x$  处可导,

即 
$$(f'(x))' = \lim_{\Delta x \rightarrow 0} \frac{f'(x + \Delta x) - f'(x)}{\Delta x}$$

存在, 则称  $(f'(x))'$  为函数  $f(x)$  在点  $x$  处的**二阶**

**导数**, 记为  $f''(x), y'', \frac{d^2 y}{dx^2}$  或  $\frac{d^2 f(x)}{dx^2}$ .



## 高阶导数的定义

类似地，二阶导数的导数称为**三阶导数**，记为

$$f'''(x), y''', \frac{d^3 y}{dx^3}.$$

一般地， $f(x)$ 的  $n-1$  阶导数的导数称为  $f(x)$ 的  **$n$  阶导数**，记为

$$f^{(n)}(x), y^{(n)}, \frac{d^n y}{dx^n} \text{ 或 } \frac{d^n f(x)}{dx^n}.$$

**注：** 二阶和二阶以上的导数统称为**高阶导数**。

相应地， $f(x)$  称为**零阶导数**； $f'(x)$  称为**一阶导数**。



## 计算高阶导数的方法

1. **直接法**: 由高阶导数的定义逐步求高阶导数.

例如,  $y = ax + b$ , 则有

$$y' = a, y'' = 0, \Delta, y^{(n)} = 0 (n \geq 3).$$

$y = e^x$ , 则有

$$y' = e^x, y'' = e^x, y''' = e^x, \Delta, y^{(n)} = e^x (n \geq 3).$$

一般地,  $y^{(n)} = e^x$

2. **间接法**: 利用已知的高阶导数公式, 通过导数的四则运算, 变量代换等方法, 求出  $n$  阶导数.



## ❖ 高阶导数求法举例

**例1** 设  $y = f(x) = \arctan x$ , 求  $f'''(0)$ .

**解** 
$$y' = \frac{1}{1+x^2},$$

$$y'' = \left( \frac{1}{1+x^2} \right)' = \frac{-2x}{(1+x^2)^2},$$

$$y''' = \left( \frac{-2x}{(1+x^2)^2} \right)' = \frac{2(3x^2-1)}{(1+x^2)^3},$$

$$f'''(0) = \frac{2(3x^2-1)}{(1+x^2)^3} \Big|_{x=0} = -2.$$



**例2** 设  $y = x^a (a \in R)$ , 求  $y^{(n)}$ .

**解**  $y' = ax^{a-1}$ ,

$$y'' = (ax^{a-1})' = a(a-1)x^{a-2},$$

$$y''' = (a(a-1)x^{a-2})' = a(a-1)(a-2)x^{a-3},$$

$$y^{(n)} = a(a-1)\underset{\text{L}}{\text{L}} \underset{\text{L}}{\text{L}} (a-n+1)x^{a-n} (n \geq 1),$$

若  $a$  为自然数  $n$ , 则

$$y^{(n)} = (x^n)^{(n)} = n!, \quad y^{(n+1)} = (n!)' = 0.$$

**注:** 求  $n$  阶导数时, 求出1-3或4阶后, 不要急于合并, 分析结果的规律性, 写出  $n$  阶导数 (利用数学归纳法).



**例3** 设  $y = \ln(1+x)$ , 求  $y^{(n)}$ .

**解** 
$$y' = \frac{1}{1+x},$$

$$y'' = -\frac{1}{(1+x)^2},$$

$$y''' = \frac{2!}{(1+x)^3},$$

$$y^{(4)} = -\frac{3!}{(1+x)^4},$$

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$$y^{(n)} = (-1)^{n-1} \frac{(n-1)!}{(1+x)^n} \quad (n \geq 1, 0! = 1).$$



补例  $(e^{ax})^{(n)} = a^n e^{ax}$

$$(e^{ax})' = ae^{ax}, (e^{ax})'' = (ae^{ax})' = a^2 e^{ax}, \Delta$$

补例  $(\frac{1}{x+a})^{(n)} = (-1)^n \frac{n!}{(x+a)^{n+1}}$

$$(\frac{1}{x+a})' = -(x+a)^{-2},$$

$$(\frac{1}{x+a})'' = [-(x+a)^{-2}]' = (-2)(-1)(x+a)^{-3}, \Delta$$

补例  $[\ln(x+a)]^{(n)} = (-1)^{(n-1)} \frac{(n-1)!}{(x+a)^n}$

$$[\ln(x+a)]' = \frac{1}{x+a}, [\ln(x+a)]^{(n)} = (\frac{1}{x+a})^{(n-1)},$$





例4  $y = \sin x, (y)^{(2006)} = -\sin x$

$$y' = \cos x,$$

$$y'' = -\sin x,$$

$$y''' = -\cos x,$$

$$y^{(4)} = \sin x,$$

$$(\sin x)^{(n)} =$$



$$y' = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

$$y'' = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + 2 \cdot \frac{\pi}{2}\right),$$

$$y''' = \cos\left(x + 2 \cdot \frac{\pi}{2}\right) = \sin\left(x + 2 \cdot \frac{\pi}{2} + \frac{\pi}{2}\right) = \sin\left(x + 3 \cdot \frac{\pi}{2}\right),$$

一般地, 可得

$$y^{(n)} = \sin\left(x + n \cdot \frac{\pi}{2}\right),$$

$$(\sin x)^{(n)} = \sin\left(x + n \cdot \frac{\pi}{2}\right)$$

类似可得

$$(\cos x)^{(n)} = \cos\left(x + n \cdot \frac{\pi}{2}\right)$$



**例5** 求  $y = \sin kx$  , 求  $y^{(n)}$  .

**解**  $y' = k \cos kx = k \sin\left(kx + \frac{\pi}{2}\right),$

$$y'' = (y')' = k^2 \cos\left(kx + \frac{\pi}{2}\right)$$
$$= k^2 \sin\left(kx + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= k^2 \sin\left(kx + 2 \cdot \frac{\pi}{2}\right),$$

$$y''' = (y'')' = k^3 \cos\left(kx + 2 \cdot \frac{\pi}{2}\right)$$



$$\begin{aligned}y' &= k \cos kx = k \sin\left(kx + \frac{\pi}{2}\right), \\y'' &= (y')' = k^2 \sin\left(kx + 2 \cdot \frac{\pi}{2}\right), \\y''' &= (y'')' = k^3 \cos\left(kx + 2 \cdot \frac{\pi}{2}\right)\end{aligned}$$

$$y^{(n)} = k^n \sin\left(kx + n \cdot \frac{\pi}{2}\right),$$

即  $(\sin kx)^{(n)} = k^n \sin\left(kx + n \cdot \frac{\pi}{2}\right).$

同理可得  $(\cos kx)^{(n)} = k^n \cos\left(kx + n \cdot \frac{\pi}{2}\right).$



## • 几个 $n$ 阶导数公式

$$(e^{ax})^{(n)} = a^n e^{ax}$$

$$\left(\frac{1}{x+a}\right)^{(n)} = (-1)^n \frac{n!}{(x+a)^{n+1}}$$

$$[\ln(x+a)]^{(n)} = (-1)^{(n-1)} \frac{(n-1)!}{(x+a)^n}$$

$$(\sin x)^{(n)} = \sin\left(x + n \cdot \frac{\pi}{2}\right)$$

$$(\cos x)^{(n)} = \cos\left(x + n \cdot \frac{\pi}{2}\right)$$



## • $n$ 阶导数的线性性质

$$(u \pm v)^{(n)} = u^{(n)} \pm v^{(n)}; \quad (Cu)^{(n)} = Cu^{(n)}.$$

例6 设  $y = \frac{1}{x^2 - 1}$ , 求  $y^{(5)}$ .

解  $\ominus$   $y = \frac{1}{x^2 - 1} = \frac{1}{2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right)$

$$\begin{aligned} \therefore y^{(5)} &= \frac{1}{2} \left[ \frac{-5!}{(x-1)^6} - \frac{-5!}{(x+1)^6} \right] \\ &= 60 \left[ \frac{1}{(x+1)^6} - \frac{1}{(x-1)^6} \right] \end{aligned}$$



## 思考题1

$$y = \frac{x^{2006}}{x^2 - 1}, \text{求 } y^{(2006)}.$$

解题提示:

$$a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + \Lambda + a + 1)$$

$$\frac{x^{2006} - 1}{x^2 - 1} = x^{2004} + x^{2002} + \Lambda + x^2 + 1$$

$$\frac{1}{x^2 - 1} = \frac{1}{2} \left( \frac{1}{x - 1} - \frac{1}{x + 1} \right)$$



## 思考题2

设  $g'(x)$  连续, 且  $f(x) = (x-a)^2 g(x)$ , 求  $f''(a)$ .

解  $\ominus g(x)$  可导

$$\therefore f'(x) = 2(x-a)g(x) + (x-a)^2 g'(x)$$

$\ominus g''(x)$  不一定存在, 故用定义求  $f''(a)$

$$f''(a) = \lim_{x \rightarrow a} \frac{f'(x) - f'(a)}{x - a} \quad f'(a) = 0$$

$$= \lim_{x \rightarrow a} \frac{f'(x)}{x - a} = \lim_{x \rightarrow a} [2g(x) + (x-a)g'(x)] = 2g(a)$$





### 思考题3

已知  $f^{(n)}(x) = g(x)$ , 则  $f^{(n)}(ax + b) = g(ax + b)$ ,

$$[f(ax + b)]^{(n)} = a^n f^{(n)}(ax + b) = a^n g(ax + b).$$

**推导:**  $[f(ax + b)]' = f'(ax + b) \cdot (ax + b)' = af'(ax + b)$ ,

$$\begin{aligned} [f(ax + b)]'' &= [af'(ax + b)]' \\ &= af''(ax + b) \cdot (ax + b)' = a^2 f''(ax + b), \end{aligned}$$

$\Lambda \Lambda$

$$[f(ax + b)]^{(n)} = a^n f^{(n)}(ax + b).$$

**提示:**  $f(x) = e^x$ ,  $f^{(n)}(x) = e^x$ , 则

$$f^{(n)}(ax) = e^{ax}, \quad [f(ax)]^{(n)} = (e^{ax})^{(n)} = a^n e^{ax},$$

即  $[f(ax)]^{(n)} = a^n f^{(n)}(ax)$



### 思考题3

已知  $f^{(n)}(x) = g(x)$ , 则  $f^{(n)}(ax + b) = g(ax + b)$ ,

$$[f(ax + b)]^{(n)} = a^n f^{(n)}(ax + b) = a^n g(ax + b).$$

例如 由  $(e^x)^{(n)} = e^x$ ,

得  $(e^{ax})^{(n)} = a^n e^{ax}$ .

由  $(\sin x)^{(n)} = \sin(x + n \cdot \frac{\pi}{2})$ ,

得  $[\sin(ax + b)]^{(n)} = a^n \sin(ax + b + n \cdot \frac{\pi}{2})$ .

由  $(\cos x)^{(n)} = \cos(x + n \cdot \frac{\pi}{2})$ ,

得  $[\cos(ax + b)]^{(n)} = a^n \cos(ax + b + n \cdot \frac{\pi}{2})$ .



## 思考题4

设  $y = \sin^6 x + \cos^6 x$ , 求  $y^{(n)}$ .

解

$$\begin{aligned}y &= (\sin^2 x)^3 + (\cos^2 x)^3 \\&= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x) \\&= (\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cos^2 x \\&= 1 - \frac{3}{4} \sin^2 2x = 1 - \frac{3}{4} \cdot \frac{1 - \cos 4x}{2} \\&= \frac{5}{8} + \frac{3}{8} \cos 4x \\ \therefore y^{(n)} &= \frac{3}{8} \cdot 4^n \cdot \cos\left(4x + n \cdot \frac{\pi}{2}\right).\end{aligned}$$



# 作业

习题2-4 (P106):

1.单号

6.(2)